

Galois Theory for H -extensions and H -coextensions

Dorota Marciniak

National Institute of Telecommunications, Warsaw, Poland
DOROFIA@GMAIL.COM

Marcin Szamotulski*

IST, Lisbon, Portugal
MSZAMOT@GMAIL.COM

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Abstract

We show that there exists a Galois correspondence between subalgebras of an H -comodule algebra A over a base ring R and generalised quotients of a Hopf algebra H . We also show that Q -Galois subextensions are closed elements of the constructed Galois connection. Then we consider the theory of coextensions of H -module coalgebras. We construct Galois theory for them and we prove that H -Galois coextensions are closed. We apply the obtained results to the Hopf algebra itself and we show a simple proof that there is a bijection correspondence between right ideal coideals of H and its left coideal subalgebras when H is finite dimensional. Furthermore we formulate necessary and sufficient conditions when the Galois correspondence is a bijection for arbitrary Hopf algebras. We also present new conditions for closedness of subalgebras and generalised quotients when A is a crossed product.

1 Introduction

Hopf-Galois extensions have roots in the approach of Chase et al. [1965] who generalised the *classical Galois Theory* for field extensions to commutative rings. In the following paper Chase and Sweedler [1969] extended these ideas to coactions of Hopf algebras on *commutative algebras* over rings. The general definition of a Hopf-Galois extension was first introduced by Kreimer and Takeuchi [1981]. Under the assumption that H is finite dimensional their definition is equivalent to the now days standard

Definition 1.1

An H -extension $A/A^{co H}$ is called H -Hopf-Galois extension (H -Galois extension, for short) if the canonical map of right H -comodules and left A -modules:

$$can : A \otimes_{A^{co H}} A \longrightarrow A \otimes H, \quad a \otimes b \longmapsto ab_{(0)} \otimes b_{(1)} \quad (1)$$

is an isomorphism, $A^{co H} := \{a \in A : a_{(0)} \otimes a_{(1)} = a \otimes 1_H\}$.

*Second address: National Institute of Telecommunications, Warsaw, Poland.

A breakthrough was made by extending the results of Chase and Sweedler [1969] to noncommutative setting by van Oystaeyen and Zhang. They construct Galois correspondence for Hopf-Galois extensions which has particularly good properties for field extensions (see [van Oystaeyen and Zhang, 1994, Thm 4.4 and Cor. 4.6]). Van Oystaeyen and Zhang introduce a remarkable construction of an *associated Hopf algebra* to an H -extension $A/A^{co H}$, where A as well as H are supposed to be commutative ([van Oystaeyen and Zhang, 1994, Sec. 3], for noncommutative generalisation see: Schauenburg [1996, 1998]). We will denote this Hopf algebra by $L(H, A)$. Schauenburg [1998, Prop. 3.2] generalises the van Oystaeyen and Zhang correspondence (see also [Schauenburg, 1996, Thm 6.4]) to Galois connection between generalised quotients of the associated Hopf algebra $L(H, A)$ (i.e. quotients by right ideal coideals) and subextensions of a faithfully flat H -Hopf Galois extension of the base ring. In this work we construct a Galois correspondence without the assumption that the coinvariants subalgebra is commutative and we also drop the Hopf-Galois assumption (Theorem 4.1). Instead of Hopf theoretic approach of van Oystaeyen, Zhang and Schauenburg we propose to look from lattice theoretic perspective. Using an existence theorem for Galois connections we show that if the comodule algebra A is flat over R and the functor $A \otimes_R -$ preserves infinite intersections then there exists a Galois correspondence between subalgebras of A and generalised quotients of the Hopf algebra H . It turns out that such modules are exactly the Mittag-Leffler modules (Corollary 3.7). We consider modules with intersection property in Section 3, where we also give examples of flat and faithfully flat modules which fail to have it. Then we discuss Galois closedness of generalised quotients and subalgebras. We show that if a generalised quotient Q is such that $A/A^{co Q}$ is Q -Galois then it is necessarily closed assuming that $A/A^{co H}$ has epimorphic canonical map (Corollary 4.3). Later we prove that this is also a necessary condition for Galois closedness if $A = H$ or, more generally, if $A/A^{co H}$ is a crossed product, H is flat and $A^{co H}$ is a flat Mittag-Leffler R -module (Theorem 7.1). We also consider the dual case: of H -module coalgebras, which later gives us a simple proof of bijective correspondence between generalised quotients and left ideal subalgebras of H if it is finite dimensional (Theorem 6.1). This Takeuchi correspondence, dropping the assumptions of faithfully (co)flatness in [Schauenburg, 1998, Thm. 3.10], was proved by Skryabin [2007], who showed that finite dimensional Hopf algebra is free over any its left coideal subalgebra. Our proof avoids using this result. We also characterise closed elements of this Galois correspondence in general case (Theorem 6.2). As we already mentioned, we show that a generalised quotient Q is closed if and only if $H/H^{co Q}$ is a Q -Galois extension. Furthermore, we show that a left coideal subalgebra K is closed if and only if $H \rightarrow H/K^+ H$ is a K -Galois coextension (see Definition 5.2). This gives an answer to the question when the bijective correspondence between generalised quotients over which H is faithfully coflat and coideal subalgebra over which H is faithfully flat holds without (co)flatness assumptions. In the last section we extend the characterisation of closed subalgebras and closed generalised quotients to crossed products.

2 Preliminaries

A partially ordered set, or poset for short, is a set P together with a reflexive, transitive and anti-symmetric relation \leq . The dual poset to a poset P we will denote by P^{op} . If

the partial order of P is denoted by \leq then the partial order \leq^{op} of P^{op} is defined by $p_1 \leq^{op} p_2 \Leftrightarrow p_2 \leq p_1$. Infima in P will be denoted by \bigwedge , i.e. $\inf_{i \in I} p_i = \bigwedge_{i \in I} p_i$, for $p_i \in P$. We will write \bigvee for suprema in P . A poset which has all finite infima and suprema is called a lattice. It is called complete if arbitrary infima and suprema exist.

Definition 2.1

Let P and Q be two posets. Then a Galois connection is a pair (F, G) of antimonotonic maps: $F : P \rightrightarrows Q : G$ such that $\forall_{p \in P, q \in Q} \quad GFp \geq p$ and $FG(q) \geq q$. An element p of P (respectively $q \in Q$) is called closed if and only if $GFp = p$ ($FGq = q$). The set of closed elements of P (Q) we will denote by \overline{P} (\overline{Q} respectively).

Proposition 2.2 (Davey and Priestley [2002])

Let (F, G) be a Galois connection between the poset P and Q . Then:

- (1) $\overline{P} = G(Q)$ and $\overline{Q} = F(P)$,
- (2) The restrictions $F|_{\overline{P}}$ and $G|_{\overline{Q}}$ are inverse bijections of \overline{P} and \overline{Q} (F and G restricts to inverse bijections).
- (3) The map F is unique in the sense that there exists only one Galois connection of the form (F, G) , in a similar way G is unique.
- (4) The map F is mono (onto) if and only if the map G is onto (mono).
- (5) If one of the two maps F, G is an isomorphism then the second is its inverse.

If P and Q are complete lattices then so are the posets of closed elements.

Let us note that if (F, G) is a Galois connection between posets then F and G reflects all existing suprema into infima.

Theorem 2.3

Let P and Q be two posets. Let $F : P \rightarrow Q$ be an anti-monotonic map of posets. If P is complete then there exists Galois connection (F, G) if and only if F reflects all suprema.

For the proof we refer to [Davey and Priestley, 2002, Prop. 7.34].

All algebras, coalgebras, Hopf algebras, etc. if not otherwise stated, are assumed to be over a commutative ring R . The unadorned tensor product \otimes will denote the tensor product over the base ring R . We refer to or Brzeziński and Wisbauer [2003] for the basic definitions. Let us recall that a coideal of an R -coalgebra C is kernel of a coalgebra epimorphism with source C . For a coalgebra C over a ring R the set of all coideals of the coalgebra C forms a complete lattice with inclusion as the order relation. It will be denoted by $\text{cold}(C)$. The complete poset of left coideals we let denote by $\text{cold}_l(C)$. A Hopf ideal of a Hopf algebra H is a kernel of an epimorphism of Hopf algebras with source H . The set of Hopf ideals form a complete lattice which will be denoted by $\text{Id}_{\text{Hopf}}(H)$. The dual lattice we will denote by $\text{Quot}(H)$.

Definition 2.4

A generalised quotient Q of a Hopf algebra H is a quotient by a right ideal coideal. The poset of generalised quotients will be denoted by $\text{Quot}_{\text{gen}}(H)$. The order relation of $\text{Quot}_{\text{gen}}(H)$ we will denote by \succsim .

A generalised subalgebra K of a Hopf algebra H is a left coideal subalgebra. The poset of generalised subalgebras will be denoted by $\text{Sub}_{\text{gen}}(H)$.

The poset $\text{Quot}_{\text{gen}}(H)$ is dually isomorphic to the poset of right ideals coideals of H , which will be denoted as $\text{Id}_{\text{gen}}(H)$. We define only right version of generalised quotients and left version of generalised subalgebras since we will deal only with right H -comodules.

Proposition 2.5

Let H be a Hopf algebra. Then the poset $\text{Quot}_{\text{gen}}(H)$ is a complete lattice.

Proof: Note that the supremum in $\text{Id}_{\text{gen}}(H)$ is given by sum of submodules. It follows that $\text{Id}_{\text{gen}}(H)$ is a complete upper lattice, and thus it is a complete lattice, since in any complete lattice we have: $x \wedge y = \bigvee_{z \leq x \text{ and } z \leq y} z$. ■

The infimum in $\text{Id}_{\text{gen}}(H)$ is given by the formula:

$$I \wedge J = \bigoplus_{\substack{K \subseteq I \cap J \\ K \in \text{Id}_{\text{gen}}(H)}} K$$

where $I, J \in \text{Id}_{\text{gen}}(H)$. Furthermore, let us note that if H is finite dimensional (over a field) then the lattice of generalised ideals of a Hopf is both algebraic and dually algebraic.

An R -algebra A is an H -comodule algebra if A is a (coassociative and counital) H -comodule with structure maps: $\delta_A : A \rightarrow A \otimes H$ and $\text{id}_A \otimes \varepsilon : A \otimes H \rightarrow A$, which are algebra homomorphisms (with the usual algebra structure on the tensor product). We denote the subalgebra of coinvariants by $A^{\text{co}H} := \{a \in A : \delta_A(a) = a \otimes 1_H\}$. For an algebra A the poset of all subalgebras will be denoted by $\text{Sub}_{\text{Alg}}(A)$ and $\text{Sub}_{\text{Alg}}(A/B) := \{S \in \text{Sub}_{\text{Alg}}(A) : B \subseteq S\}$. Let Q be a generalised quotient of H . Then A is a Q -comodule with the structure map $\delta_Q := \text{id}_A \otimes \pi_Q \circ \delta_A$, where $\pi_Q : H \rightarrow Q$ is the projection. If Q is a Hopf quotient then this coaction turns A into a Q -comodule algebra.

3 Modules with intersection property

For a flat R -module M the tensor product functor $M \otimes -$ preserves all finite intersections (see [Brzeziński and Wisbauer, 2003, 40.16]). Furthermore, it is not hard to show that tensoring with a flat module preserves all finite limits. In this section we show that there is a large class of modules for which the tensor product functor preserves arbitrary intersections. We will also construct examples of flat and faithfully modules without this property.

Let N' be a submodule of N , $i : N' \subseteq N$, and let M be an R -module. Then the canonical image of $M \otimes N'$ in $M \otimes N$ is the image of $M \otimes N'$ under the map $\text{id}_M \otimes i$. It will be denoted by $\text{im}(M \otimes N')$.

Definition 3.1

Let M, N be R -modules, and let $(N_\alpha)_{\alpha \in I}$ be a family of submodules of an R -module N . We say that a module M has the **intersection property with respect to N** if the homomorphism:

$$\text{im}(M \otimes (\bigcap_{\alpha \in I} N_\alpha)) \longrightarrow \bigcap_{\alpha \in I} \text{im}(M \otimes N_\alpha)$$

is an isomorphism for any family of submodules $(N_\alpha)_{\alpha \in I}$. We say that M has the **intersection property** if the above condition holds for any R -module N .

Note that if M is flat then it has the intersection property if and only if the map $M \otimes (\bigcap_{\alpha \in I} N_\alpha) \rightarrow \bigcap_{\alpha \in I} (M \otimes N_\alpha)$ is an isomorphism.

Proposition 3.2

The intersection property is closed under direct sums.

Proof: Let $X = \oplus_{i \in I} X_i$, be a direct sum of modules with intersection property. We let $\pi_i : \oplus_{i \in I} X_i \rightarrow X_i$ be the canonical projection on i -th factor and $s_i : X_i \rightarrow \oplus_{i \in I} X_i$ be the canonical section. Let $(N_\alpha)_{\alpha \in J}$ be a family of submodules of an R -module N . Then we have a split epimorphism $(\oplus_{i \in I} \pi_i) \otimes id_N$ with a right inverse $(\oplus_{i \in I} s_i) \otimes id_N$. Also for each $\alpha \in J$ the map $(\oplus_{i \in I} \pi_i) \otimes id_{N_\alpha}$ is a split epimorphism with right inverse $(\oplus_{i \in I} s_i) \otimes id_{N_\alpha}$. Furthermore, we have a family of split epimorphisms, which sections are jointly surjective:

$$\begin{array}{ccc} & \xleftarrow{s_i \otimes id_N} & \\ \text{im}((\oplus_{i \in I} X_i) \otimes N_\alpha) & \xrightarrow{\pi_i \otimes id_N} & \text{im}(X_i \otimes N_\alpha) \end{array}$$

They induce the following family of projections with jointly surjective sections:

$$\begin{array}{ccc} & \xleftarrow{s_i \otimes id_N} & \\ \bigcap_{\alpha \in J} \text{im}((\oplus_{i \in I} X_i) \otimes N_\alpha) & \xrightarrow{\pi_i \otimes id_N} & \bigcap_{\alpha \in J} \text{im}(X_i \otimes N_\alpha) \end{array}$$

For this let $x \in \bigcap_{\alpha \in J} \text{im}((\oplus_{i \in I} X_i) \otimes N_\alpha)$. Then, for each $\alpha \in J$, there exists $y_\alpha \in \oplus_{i \in I} \text{im}(X_i \otimes N_\alpha)$ such that $(\oplus_{i \in I} s_i \otimes id_N)(y_\alpha) = \sum_{i \in I} s_i \otimes id_N(y_\alpha) = x$. Since $\oplus_{i \in I} s_i \otimes id_N$ is a monomorphism we get $y = y_\alpha \in \bigcap_{\alpha \in J} \oplus_{i \in I} \text{im}(X_i \otimes N_\alpha)$ for all $\alpha \in J$. It follows that:

$$\bigcap \text{im}((\oplus_{i \in I} X_i) \otimes N_\alpha) \xrightarrow{\oplus_{i \in I} \pi_i \otimes id_N} \oplus_{i \in I} \bigcap_{\alpha \in J} \text{im}(X_i \otimes N_\alpha)$$

is an isomorphism with inverse $\oplus_{i \in I} s_i \otimes id_N$. Now the proposition will follow from the commutativity of the diagram:

$$\begin{array}{ccc} \text{im}((\oplus_{i \in I} X_i) \otimes (\bigcap_{\alpha \in J} N_\alpha)) & \xrightarrow{\quad} & \bigcap_{\alpha \in J} \text{im}((\oplus_{i \in I} X_i) \otimes N_\alpha) \\ \simeq \downarrow & & \uparrow \simeq \\ \text{im}(\oplus_{i \in I} (X_i \otimes (\bigcap_{\alpha \in J} N_\alpha))) & \xrightarrow{\quad} \oplus_{i \in I} \text{im}(X_i \otimes (\bigcap_{\alpha \in J} N_\alpha)) \xrightarrow{\simeq} \oplus_{i \in I} \bigcap_{\alpha \in J} \text{im}(X_i \otimes N_\alpha) \end{array} \quad (2)$$

We will go around this diagram from the top left corner to the top right one and prove that all the maps on the way are isomorphisms. The first map is an isomorphism since tensor product commutes with colimits. Clearly the second map is an isomorphism as well. The bottom right arrow in (2) is an isomorphism since all X_i ($i \in I$) have the intersection property and we already showed that the last homomorphism is an isomorphism. ■

Proposition 3.3

The intersection property is stable under taking direct summands.

Proof: Let M be a direct summand in a module P which has intersection property. Let M' be the complement of M in P . Then we have a chain of isomorphisms:

$$\begin{aligned}
 \text{im}\left(M \otimes \bigcap_{\alpha \in J} N_{\alpha}\right) \oplus \text{im}\left(M' \otimes \bigcap_{\alpha \in J} N_{\alpha}\right) &\simeq \text{im}\left((M \oplus M') \otimes \left(\bigcap_{\alpha \in J} N_{\alpha}\right)\right) \\
 &= \bigcap_{\alpha \in J} \text{im}\left((M \oplus M') \otimes N_{\alpha}\right) \\
 &\simeq \bigcap_{\alpha \in J} \text{im}\left(M \otimes N_{\alpha} \oplus M' \otimes N_{\alpha}\right) \\
 &\simeq \bigcap_{\alpha \in J} \text{im}\left(M \otimes N_{\alpha}\right) \oplus \bigcap_{\alpha \in J} \text{im}\left(M' \otimes N_{\alpha}\right)
 \end{aligned}$$

Since every isomorphism in above diagram commutes with projection onto the first and second factor the composition also does, and thus it is direct sum of the two natural maps: $\text{im}\left(M \otimes \bigcap_{\alpha \in J} N_{\alpha}\right) \rightarrow \bigcap_{\alpha \in J} \text{im}\left(M \otimes N_{\alpha}\right)$, $\text{im}\left(M' \otimes \bigcap_{\alpha \in J} N_{\alpha}\right) \rightarrow \bigcap_{\alpha \in J} \text{im}\left(M' \otimes N_{\alpha}\right)$. It follows that both maps are isomorphisms, hence both M and M' have the intersection property. \blacksquare

Since pure projective modules are direct summands in sums of finitely presented modules and tensor product with finitely presented modules preserves all limits we get that:

Corollary 3.4

Every pure projective module has the intersection property.

Proposition 3.5

Let M be a Mittag–Leffler R -module. Then M has the intersection property.

We wish to thank Christian Lomp for presenting us this result and pointing us to [Herbar and Trlifaj \[2009\]](#). Note that for the case of flat Mittag–Leffler modules the above result follows from [\[Raynaud and Gruson, 1971, Cor. 2.1.7\]](#).

Proof: Let N_{α} for $\alpha \in I$ be a family of submodule an R -module N . Let us consider the following diagram:

$$\begin{array}{ccccc}
 0 & \longrightarrow & M \otimes (N / \bigcap_{\alpha \in I} N_{\alpha}) & \xrightarrow{i} & M \otimes (\prod_{\alpha \in I} N / N_{\alpha}) \\
 & & \downarrow G & & \downarrow f \\
 & & & & \prod_{\alpha \in I} (M \otimes (N / N_{\alpha})) \\
 & & & & \downarrow g \\
 0 & \longrightarrow & (M \otimes N) / \bigcap_{\alpha \in I} \text{im}(M \otimes N_{\alpha}) & \xrightarrow{j} & \prod_{\alpha \in I} (M \otimes N) / \text{im}(M \otimes N_{\alpha})
 \end{array}$$

Where i and j are the canonical embeddings: $i(m \otimes (n + \bigcap_{\alpha \in I} N_{\alpha})) := m \otimes (n + N_{\alpha})_{\alpha \in I}$ and $j((m \otimes n) + \bigcap_{\alpha \in I} \text{im}(M \otimes N_{\alpha})) = (m \otimes n + \text{im}(M \otimes N_{\alpha}))_{\alpha \in I}$ for $m \in M$ and $n \in N$. While f sends $m \otimes (n_{\alpha} + N_{\alpha})_{\alpha \in I}$ to $(m \otimes (n_{\alpha} + N_{\alpha}))_{\alpha \in I}$ and g is the canonical isomorphism. Note that $\text{im}(gfi) \subseteq \text{im}(j)$ and hence if M is Mittag–Leffler, then $G := gfi$ can be considered an embedding $G : M \otimes (N / \bigcap_{\alpha \in I} N_{\alpha}) \rightarrow (M \otimes N) / \bigcap_{\alpha \in I} \text{im}(M \otimes N_{\alpha})$. Hence we get the exact diagram:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{im} \left(M \otimes \left(\bigcap_{\alpha \in I} N_{\alpha} \right) \right) & \longrightarrow & M \otimes N & \longrightarrow & M \otimes (N / \bigcap_{\alpha \in I} N_{\alpha}) \longrightarrow 0 \\
& & \downarrow H & & \downarrow = & & \downarrow G \\
0 & \longrightarrow & \bigcap_{\alpha \in I} \text{im} (M \otimes N_{\alpha}) & \longrightarrow & M \otimes N & \longrightarrow & (M \otimes N) / \bigcap_{\alpha \in I} \text{im} (M \otimes N_{\alpha}) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \text{coker}(H) & & 0 & & \text{coker}(G)
\end{array}$$

Where H is the canonical embedding of $\text{im} \left(M \otimes \left(\bigcap_{\alpha \in I} N_{\alpha} \right) \right)$ into $\text{im} \left(\bigcap_{\alpha \in I} (M \otimes N_{\alpha}) \right)$ as submodules of $M \otimes N$. By the snake lemma we get the following short exact sequence:

$$0 = \ker(G) \longrightarrow \text{coker}(H) \longrightarrow 0$$

Thus the monomorphism H is onto. ■

Since a module is flat Mittag-Leffler if and only if it is \aleph_1 -projective (see [Herbar and Trlifaj, 2009, Thm. 2.9]) we get the following

Corollary 3.6

Any \aleph_1 -projective module has the intersection property.

Now using [Raynaud and Gruson, 1971, Prop. 2.1.8] we obtain:

Corollary 3.7

A flat module has the intersection property if and only if it satisfies the Mittag-Leffler condition.

EXAMPLE 3.8 ¹ Let p be a prime ideal of \mathbb{Z} . Then $\bigcap_i p^i = \{0\}$. We let \mathbb{Z}_p denote the ring of fractions of \mathbb{Z} with respect to p . The \mathbb{Z} -module \mathbb{Z}_p is flat, and $\mathbb{Z}_p \otimes_{\mathbb{Z}} \bigcap_i p^i = \{0\}$. From the other side $\bigcap_i \mathbb{Z}_p \otimes_{\mathbb{Z}} p^i \cong \mathbb{Z}_p$. By similar argument $\mathbb{Q} \otimes_{\mathbb{Z}} -$ doesn't possess the intersection property, even though it is flat over \mathbb{Z} . The problem is that, the intersection property is stable under arbitrary sums but not under cokernels. Now it is easy to construct a faithfully flat module which does not have the intersection property. The \mathbb{Z} -modules $\mathbb{Z} \oplus \mathbb{Z}_p$ and $\mathbb{Z} \oplus \mathbb{Q}$ are the examples.

In the proof of Proposition 3.4 we showed that the intersection property is stable under split exact sequences. However, the above examples show that the intersection property is not stable under pure (exact) sequences [Lam, 1999, Def. 4.83], i.e. whenever $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is a pure exact sequence and M has the intersection property then M'' might not have it. It is well known that if M'' is flat then $M' \subseteq M$ is pure [Lam, 1999, Thm 4.85], hence 3.8 is indeed a source of examples. However, we can show the following proposition:

Proposition 3.9

Let M' be a pure submodule of a module M with the intersection property. Then M' has the intersection property.

¹We wish to thank S.Papadakis for discussions which led to these examples.

This property is shared by the classes of flat or Mittag–Leffler modules.

Proof: We have commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{im} (M \otimes (\bigcap_{\alpha \in I} N_{\alpha})) & \longrightarrow & M \otimes N & \xrightarrow{g} & M \otimes (N / \bigcap_{\alpha \in I} N_{\alpha}) \longrightarrow 0 \\
 & & \nearrow & & \nearrow & & \nearrow \\
 0 & \longrightarrow & \text{im} (M' \otimes (\bigcap_{\alpha \in I} N_{\alpha})) & \xrightarrow{H} & M' \otimes N & \xrightarrow{f} & M' \otimes (N / \bigcap_{\alpha \in I} N_{\alpha}) \longrightarrow 0 \\
 & & \downarrow H' & & \downarrow \parallel & & \downarrow \parallel \\
 & & 0 & \longrightarrow & \bigcap_{\alpha \in I} \text{im} (M \otimes N_{\alpha}) & \longrightarrow & M \otimes N \longrightarrow (M \otimes N) / \bigcap_{\alpha \in I} \text{im} (M \otimes N_{\alpha}) \longrightarrow 0 \\
 & & \downarrow & & \downarrow \parallel & & \downarrow G' \\
 0 & \longrightarrow & \bigcap_{\alpha \in I} \text{im} (M' \otimes N_{\alpha}) & \longrightarrow & M' \otimes N & \longrightarrow & (M' \otimes N) / \bigcap_{\alpha \in I} \text{im} (M' \otimes N_{\alpha}) \longrightarrow 0
 \end{array}$$

It easily follows that H' is a monomorphism. Let $x \in \bigcap_{\alpha \in I} \text{im} (M' \otimes N_{\alpha})$. To prove that x is in the image of H' it is enough to show that it goes to 0 under f . Now since H is an isomorphism it goes to 0 under g and thus it belongs to the kernel of f . ■

Theorem 3.10

Every flat R -module has the intersection property (or equivalently is Mittag–Leffler) if and only if for any exact sequence:

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

with M' , M projective, M'' flat and any family of submodules $(N_{\alpha})_{\alpha \in I}$ of an R -module N the sequence:

$$0 \rightarrow \bigcap_{\alpha} (M' \otimes N_{\alpha}) \rightarrow \bigcap_{\alpha} (M \otimes N_{\alpha}) \rightarrow \bigcap_{\alpha} (M'' \otimes N_{\alpha}) \rightarrow 0 \quad (3)$$

is exact.

Proof: Every flat module is a colimit of projective modules. Any colimit of projective modules can be computed as a cokernel of a map between projective modules, by [Mac Lane, 1998, Thm 1, Chap. V, §2]. So let M'' be a flat module and let

$$\mathcal{E} : 0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

be exact, where M' and M are projective modules, hence they satisfy the intersection property. The extension \mathcal{E} is pure, since M'' is flat [Lam, 1999, Thm 4.85]. Let $(N_{\alpha})_{\alpha \in I}$ be a family of submodules of an R -module N . We have a commutative diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & M' \otimes \bigcap_{\alpha} N_{\alpha} & \longrightarrow & M \otimes \bigcap_{\alpha} N_{\alpha} & \longrightarrow & M'' \otimes \bigcap_{\alpha} N_{\alpha} \longrightarrow 0 \\
 & & \downarrow \wr & & \downarrow \wr & & \downarrow f \\
 0 & \longrightarrow & \bigcap_{\alpha} (M' \otimes N_{\alpha}) & \longrightarrow & \bigcap_{\alpha} (M \otimes N_{\alpha}) & \longrightarrow & \bigcap_{\alpha} (M'' \otimes N_{\alpha}) \longrightarrow 0
 \end{array}$$

The upper row is exact by purity of $M' \subseteq M$, thus the lower row is exact if and only if the canonical map f is an isomorphism. \blacksquare

The exactness of (3) is rather difficult to obtain but it might be useful once we know that every flat module is Mittag–Leffler.

4 Galois Theory for Hopf-Galois extensions

Now we can prove existence of the Galois correspondence for comodule algebras.

Theorem 4.1

Let A/B be a H -comodule algebra over a ring R such that A is a flat Mittag–Leffler module. Then there exists a Galois connection:

$$\text{Sub}_{\text{Alg}}(A) \xrightleftharpoons[\varphi]{\psi} \text{Quot}_{\text{gen}}(H) \quad (4)$$

where $\varphi(Q) := A^{\text{co}Q}$ and ψ is given by the following formula: $\psi(S) = \bigvee \{Q \in \text{Quot}_{\text{gen}}(H) : S \subseteq A^{\text{co}Q}\}$.

If B is commutative and H is projective over the base ring then A/B is projective ($B \otimes H$)-Hopf Galois extension by [Kreimer and Takeuchi, 1981, Thm 1.7] and thus the above theorem applies as well as [Schauenburg, 1998, Prop. 3.2].

Proof: We shall show that φ reflects all suprema: $A^{\text{co} \bigvee_{i \in I} Q_i} = \bigcap_{i \in I} A^{\text{co} Q_i}$. From the set of inequalities: $\bigvee_{i \in I} Q_i \geq Q_j$ ($\forall j \in I$) it follows that $A^{\text{co} \bigvee_{i \in I} Q_i} \subseteq \bigcap_{i \in I} A^{\text{co} Q_i}$. Let us fix an element $a \in \bigcap_{i \in I} A^{\text{co} Q_i}$. We let I_i denote the coideal and right ideal such that $Q_i = H/I_i$. We identify $A \otimes I_i$ with a submodule of $A \otimes H$, what can be done under the assumption that A is flat over R . We want to show that:

$$\forall_{i \in I} a \in A^{\text{co} Q_i} \Leftrightarrow \forall_{i \in I} \delta(a) - a \otimes 1 \in A \otimes I_i \Leftrightarrow \delta(a) - a \otimes 1 \in A \otimes \bigcap_{i \in I} I_i \Leftrightarrow a \in A^{\text{co} \bigvee_{i \in I} Q_i}$$

The first equivalence is clear, the second follows from the equality: $\bigcap_{i \in I} A \otimes I_i = A \otimes \bigcap_{i \in I} I_i$ which holds since flat Mittag–Leffler modules have the intersection property. It remains to show that if $\delta(a) - a \otimes 1 \in A \otimes \bigcap_{i \in I} I_i$ then $\delta(a) - a \otimes 1 \in A \otimes \bigwedge_{i \in I} I_i$. Then it follows that $\delta(a) - a \otimes 1 \in A \otimes \bigcap_{i \in I} I_i \Leftrightarrow a \in A^{\text{co} \bigvee_{i \in I} Q_i}$. We proceed in three steps: we first prove this for H , then for $A \otimes H$ and finally for a general H -comodule algebra A . For $A = H$ this follows from existence of the Galois connection:

$$\begin{array}{ccc} & K \xrightarrow{\quad} H/HK^+ & \\ \text{Sub}_{\text{gen}}(H) & \xrightleftharpoons[H^{\text{co}Q} \longleftarrow \text{id}_Q]{} & \text{Quot}_{\text{gen}}(H) \end{array}$$

where $\text{Sub}_{\text{gen}}(H)$ is the lattice of right coideal subalgebras of H . Note that the infimum in both $\text{Sub}_{\text{gen}}(H)$ and $\text{Sub}_{\text{Alg}}(H)$ is given by intersection thus the above adjunction extends to $\text{Sub}_{\text{Alg}}(H) \rightleftarrows \text{Quot}_{\text{gen}}(H)$. Now the proof for $A \otimes H$: let $x = \sum_{k=1}^n a_k \otimes h_k \in A \otimes H$ be such that $\sum_{k=1}^n a_k \otimes \Delta(h_k) - \sum_{k=1}^n a_k \otimes h_k \otimes 1_H \in A \otimes H \otimes \bigcap_{i \in I} I_i$. Since A is flat Mittag–Leffler module, every finitely generated submodule of A is contained in a projective submodule [Herbar and Trlifaj, 2009, Thm 2.9]. Choose a dual basis for the

projective submodule A_0 which contains all the $\{a_k\}$: $\{e_j, e^j\}_{j \in J}$, where $e_j \in A_0$ and $e^j \in A_0^*$. Then for every $j \in J$ we have:

$$\sum_k e^j(a_k) \Delta(h_k) - e^j(a_k) h_k \otimes 1 \in H \otimes \bigcap_{i \in I} I_i, \text{ thus } \sum_k e^j(a_k) \Delta(h_k) - e^j(a_k) h_k \otimes 1 \in H \otimes \bigwedge_{i \in I} I_i.$$

It follows that $id_A \otimes \Delta(x) - x \otimes 1_H \in A \otimes H \otimes \bigwedge_{i \in I} I_i$. For general case, observe that if $a_{(0)} \otimes a_{(1)} - a \otimes 1 \in A \otimes \bigcap_{i \in I} I_i$ then $a_{(0)} \otimes a_{(1)} \otimes a_{(2)} - a_{(0)} \otimes a_{(1)} \otimes 1 \in A \otimes H \otimes \bigcap_{i \in I} I_i$, thus by the previous case $a_{(0)} \otimes a_{(1)} \otimes a_{(2)} - a_{(0)} \otimes a_{(1)} \otimes 1 \in A \otimes H \otimes \bigwedge_{i \in I} I_i$. Computing $id_A \otimes \varepsilon \otimes id_H$ we get $\delta(a) - a \otimes 1 \in A \otimes \bigwedge_{i \in I} I_i$. The formula $\psi(S) = \bigvee \{Q \in \text{Quot}_{gen}(H) : S \subseteq A^{coQ}\}$ is an easy consequence of the Galois connection properties. ■

4.1 Closed Elements

The main aim of this section is to characterise the closed elements of the correspondence (4).

Proposition 4.2

Let A be an H -comodule algebra over a ring R with surjective canonical map and let A be a Q_1 -Galois and a Q_2 -Galois extension where $Q_1, Q_2 \in \text{Quot}_{gen}(H)$. Then:

$$A^{coQ_1} = A^{coQ_2} \Rightarrow Q_1 = Q_2$$

Proof: Let $B = A^{coQ_1} = A^{coQ_2}$ then we have the following commutative diagram:

$$\begin{array}{ccccc} & & & A \otimes Q_1 & \\ & \nearrow \text{can}_{Q_1} & & \uparrow id \otimes \pi_1 & \\ A \otimes_B A & \longleftarrow A \otimes_{A^{coH}} A & \longrightarrow & A \otimes H & \\ & \searrow \text{can}_{Q_2} & & \downarrow id \otimes \pi_2 & \\ & & & A \otimes Q_2 & \end{array}$$

The maps can_{Q_1} and can_{Q_2} are isomorphisms. Let $f := (\text{can}_{Q_1} \circ \text{can}_{Q_2}^{-1}) \circ (id \otimes \pi_2)$. By commutativity of the above diagram, $f \circ \text{can}$ and $(id \otimes \pi_1) \circ \text{can}$ are equal. Moreover, surjectivity of can yields the equality $(\text{can}_{Q_1} \circ \text{can}_{Q_2}^{-1}) \circ (id \otimes \pi_2) = (id \otimes \pi_1)$. It follows that there exists $\pi : Q_1 \rightarrow Q_2$ such that $\text{can}_1 \circ \text{can}_2^{-1} = id \otimes \pi$ and $\pi \circ \pi_2 = \pi_1$. Furthermore, π is right H -linear and H -colinear, thus $Q_2 \succcurlyeq Q_1$. In the same way we obtain that $Q_1 \succcurlyeq Q_2$. Using antisymmetry of \succcurlyeq we get $Q_1 = Q_2$. ■

Corollary 4.3

Let A be an H -comodule algebra with epimorphic canonical map can_H such that the Galois connection (4) exists. Then $Q \in \text{Quot}_{gen}(H)$ is a closed element of Galois connection (4) if A/A^{coQ} is Q -Galois.

Proof: Fix A^{coQ} for some $Q \in \text{Quot}_{gen}(H)$ then $\varphi^{-1}(A^{coQ})$ is an upper-sublattice of $\text{Quot}_{gen}(H)$ (i.e. it is a subposet closed under finite suprema) which has the greatest element, namely $\tilde{Q} = \psi(A^{coQ})$. Moreover, \tilde{Q} is the only closed element belonging to $\varphi^{-1}(A^{coQ})$. Both $Q \leq \psi(A^{coQ})$ and the assumption that A/A^{coQ} is Q -Galois imply that $A/A^{co\tilde{Q}}$ is \tilde{Q} -Galois. To this end, we consider the commutative diagram:

$$\begin{array}{ccc}
A \otimes_B A & \xrightarrow{\text{can}_H} & A \otimes H \\
\downarrow & & \downarrow \\
A \otimes_{A^{co} \tilde{Q}} A & \xrightarrow{\text{can}_{\tilde{Q}}} & A \otimes \tilde{Q} \\
\downarrow = & & \downarrow \\
A \otimes_{A^{co} Q} A & \xrightarrow[\text{can}_Q]{\simeq} & A \otimes Q
\end{array}$$

From the lower commutative square we get that $\text{can}_{\tilde{Q}}$ is a monomorphism and from the upper commutative square we deduce that $\text{can}_{\tilde{Q}}$ is onto. Unless $\tilde{Q} = Q$ we get a contradiction with the previous proposition. ■

The above result applies also to the Galois correspondence of [Schauenburg \[1998\]](#), as it is the same as the Galois connection of Theorem 4.1 (see Proposition 2.2(3)). Since for a finite dimensional Hopf algebra H for every Q the extension $A/A^{co}Q$ is Q -Galois (see [\[Schauenburg and Schneider, 2005, Cor. 3.3\]](#)) we get the following statement.

Proposition 4.4

Let H be a finite dimensional Hopf algebra over a field k . Let A/B be an H -Hopf Galois extension. Then every $Q \in \text{Quot}_{\text{gen}}(H)$ is closed.

Now, we adopt [\[Schauenburg, 1998, Def. 3.3\]](#) to our setting.

Definition 4.5

Let C be a R -coalgebra and let $C \rightarrow \tilde{C}$ be a coalgebra quotient. Then \tilde{C} is called left (right) admissible if it is R -flat (hence faithfully flat) and C is left (right) faithfully coflat over \tilde{C} .

Let S belong to $\text{Sub}_{\text{Alg}}(A/B)$ for an H -extension A/B . Then S is called right (left) admissible if:

1. A is right (left) faithfully flat over S ,
2. for right admissibility, the composition:

$$\text{can}_S : A \otimes_S A \rightarrow A \otimes_{A^{co} \psi(S)} A \xrightarrow{\text{can}_{\psi(S)}} A \otimes \psi(S),$$

is a bijection, while for left admissibility the map:

$$\text{can}_{S^{op}} : A^{op} \otimes_{S^{op}} A^{op} \rightarrow A^{op} \otimes_{(A^{op})^{co} \psi(S^{op})} A^{op} \xrightarrow{\text{can}_{\psi(S^{op})}} A^{op} \otimes \psi(S^{op}),$$

is a bijection. These maps are well defined since $S \subseteq A^{co} \psi(S)$.

3. $\psi(S)$ is flat over R .

An element is called admissible if it is both left and right admissible.

Note that the definition is symmetric in the sense that $S \subseteq A$ is right (left) admissible if and only if $S^{op} \subseteq A^{op}$ is left (right) admissible.

REMARK 4.6 Let A/B be an H -extension, such that the Galois correspondence (4) exists. Let $S \in \text{Sub}_{\text{Alg}}(A/A^{co}H)$. Then the following holds:

- (i) if $\text{can}_S : A \otimes_S A \rightarrow A \otimes \psi(S)$ is an isomorphism and A is right or left faithfully flat over S then S is a closed element of (4);

(ii) if the natural projection $A \otimes_S A \twoheadrightarrow A \otimes_{A^{co\psi(S)}} A$ is a bijection and A is right or left faithfully flat over S then $S = A^{co\psi(S)}$, i.e. S is closed element of $\text{Sub}_{\text{Alg}}(A/B)$ in (4).

Proof: First let us prove (2). For this let us consider the commutative diagram:

$$\begin{array}{ccc} S & \subseteq & A \rightrightarrows A \otimes_S A \\ \uparrow \text{---} & & \parallel \quad \downarrow \text{---} \\ A^{co\psi(S)} & \subseteq & A \rightrightarrows A \otimes_{A^{co\psi(S)}} A \end{array}$$

The maps $A \rightarrow A \otimes_S A$ and $A \rightarrow A \otimes_{A^{co\psi(S)}} A$ send $a \in A$ to $a \otimes 1_A$ or $1_A \otimes a$ in appropriate tensor product. Since the diagram commutes and the upper row is an equaliser, by faithfully flat descent, the dashed arrow exists, i.e. $A^{co\psi(S)} \subseteq S$. We get the equality since, $S \subseteq A^{co\psi(S)}$ holds by the Galois connection property.

The first claim follows from (2) when applied to $S \subseteq A^{co\psi(S)}$. It was first observed in [Schneider, 1992, Rem. 1.2]. \blacksquare

The Remark 4.6(ii) for division algebras also follows from predual Jacobson-Bourbaki correspondence [Sweedler, 1975, Thm 2.1].

We introduce the following notation: for an algebra A , A^{op} denotes the opposite algebra which multiplication is given by $m^{op}(a \otimes b) = ba$. If H is a Hopf algebra with bijective antipode S_H then the algebra H^{op} is a Hopf algebra with the same comultiplication and the antipode $S_{H^{op}} = S_H^{-1}$. For a generalised quotient $Q = H/I \in \text{Quot}_{gen}(H)$ we put $Q^{op} := H^{op}/S_H(I) \in \text{Quot}_{gen}(H^{op})$ and we also write $I^{op} := S_H(I)$.

Theorem 4.7

Let H be a Hopf algebra with bijective antipode. Let A/B be an H -extension such that A/B is H -Galois, A^{op}/B^{op} is H^{op} -Galois and A is faithfully flat as both left and right B -module. Let us assume that the Galois connection (4) exists. Furthermore, let A be faithfully flat over R . Then the Galois connection (4) gives rise to a bijection between (left, right) admissible objects (thus (left, right) admissible objects are closed).

Note that if $B = A^{coH}$ is contained in the center of A (as it is assumed in Schauenburg [1998]) then if A/B is H -Galois then A^{op}/B^{op} is H^{op} -Galois. We have a commutative diagram:

$$\begin{array}{ccc} A^{op} \otimes_{B^{op}} A^{op} & \xrightarrow{\text{can}_{B^{op}}} & A^{op} \otimes H^{op} \\ \tau \downarrow & & \downarrow \alpha \\ A \otimes_B A & \xrightarrow{\text{can}_B} & A \otimes H \end{array}$$

where $\tau(a \otimes_{B^{op}} b) = b \otimes_B a$ is well defined since $B = B^{op}$ is contained in the center of A and $\alpha(a \otimes h) = a_{(0)} \otimes S_H^{-1}(h)a_{(1)}$ is an isomorphism with inverse $\alpha^{-1}(a \otimes h) = a_{(0)} \otimes a_{(1)}S_H(h)$.

The proof of [Schauenburg, 1998, Thm 3.6] applies (with minor changes which we spot below). Here we consider a right H -comodule algebra, while Schauenburg considers left $L(H, A)$ -comodule algebra structure. The proof relies on [Schneider, 1992, Rem. 1.2 and Thm 1.4]. We let $(\varphi^{op}, \psi^{op})$ denote the Galois connection (4) for the H^{op} -comodule algebra A^{op} . We start with a basic lemma:

Lemma 4.8

Let H be a Hopf algebra with bijective antipode. Let A/B be an H -extensions such that the Galois connection (4) exists. Then it also exists for the H^{op} -comodule algebra A^{op} . Furthermore, $\varphi^{op}(Q) = (\varphi(Q))^{op}$ and $\psi(S^{op}) = (\psi(S))^{op}$, where $S \in \text{Sub}_{Alg}(A/B)$ and $Q \in \text{Quot}_{gen}(H)$.

Proof: The first equation: $\varphi^{op}(Q) = (\varphi(Q))^{op}$ is proved in [Schauenburg, 1998, Prop. 3.5]. Both maps: $\text{Sub}_{Alg}(A/B) \ni S \mapsto S^{op} \in \text{Sub}_{Alg}(A^{op}/B^{op})$ and $\text{Quot}_{gen}(H) \ni Q \mapsto Q^{op} \in \text{Quot}_{gen}(H^{op})$ are isomorphisms of posets thus φ reflects suprema if and only if φ^{op} reflects them. It remains to show that $\psi^{op}(S^{op}) = (\psi(S))^{op}$. First, let us observe that for any set \mathcal{O} of right ideal coideals we have $\bigwedge_{I \in \mathcal{O}} I^{op} = (\bigwedge_{I \in \mathcal{O}} I)^{op}$:

$$\bigwedge_{I \in \mathcal{O}} I^{op} = \bigoplus_{\substack{J \in \text{Id}_{gen}(H^{op}) \\ J \subseteq \bigcap_{I \in \mathcal{O}} I^{op}}} J = \bigoplus_{\substack{J \in \text{Id}_{gen}(H) \\ J^{op} \subseteq (\bigcap_{I \in \mathcal{O}} I)^{op}}} J^{op} = \bigoplus_{\substack{J \in \text{Id}_{gen}(H) \\ J \subseteq \bigcap_{I \in \mathcal{O}} I}} J^{op} = \left(\bigwedge_{I \in \mathcal{O}} I \right)^{op} \quad (5)$$

Let $\overline{\psi}(S) = \ker(H \rightarrow \psi(S))$. Now using formula (4) for ψ we get:

$$\overline{\psi}(S) = \bigwedge \{I : S \subseteq A^{co H/I}\}, \quad \overline{\psi}^{op}(S^{op}) = \bigwedge \{I^{op} : S \subseteq (A^{op})^{co H^{op}/I^{op}}\}$$

since $(A^{co H/I})^{op} = (A^{op})^{co H^{op}/I^{op}}$ and $\text{Quot}_{gen}(H) \ni I \mapsto I^{op} := S_{H^{op}}(I) \in \text{Quot}_{gen}(H^{op})$ is a bijection the two above sets are in bijective correspondence: $I \mapsto I^{op}$. Now, the formula $\psi^{op}(S^{op}) = (\psi(S))^{op}$ follows from equation (5). ■

Proposition 4.9

Let H be a Hopf algebra with bijective antipode. Let A be H -extension of B such that A is H -Galois, A^{op} is H^{op} -Galois and the Galois connection (4) exists, ${}_B A$ and A_B are faithfully flat and also it is faithfully flat as an R -module. Then:

- (1) if $S \in \text{Sub}(A/B)$ is right admissible then so is $\psi(S)$ and $\varphi\psi(S) = S$,
- (2) if $Q \in \text{Quot}_{gen}(H)$ is left admissible then so is $\varphi(Q) := A^{co Q}$ and $\psi\varphi(Q) = Q$.

Proof: We first prove (2): $\psi(Q)$ is left admissible by [Schneider, 1992, Thm 1.4] applied to A^{op} . The equality $\psi^{op}\varphi^{op}(Q^{op}) = Q^{op}$ follows from Corollary 4.3. Using Lemma 4.8 we conclude $(\psi\varphi(Q))^{op} = Q^{op}$, and since $\text{Quot}_{gen}(H) \ni Q \mapsto Q^{op} \in \text{Quot}_{gen}(H^{op})$ is a bijection we get $\psi\varphi(Q) = Q$. Now, let us assume that $S \in \text{Sub}(A/B)$ is right admissible. Then by Remark 4.6(i) $S = \varphi\psi(S)$ and A/S is Q -Galois. Since, A is faithfully flat R -module and A_S is faithfully flat, it follows from the isomorphism $A \otimes_S A \cong A \otimes \psi(S)$ that $\psi(S)$ is faithfully flat R -module. It remains to show that H is faithfully coflat as right $\psi(S)$ -comodule. For this we observe, as in [Schauenburg, 1998, Prop. 3.4], that for any left $\psi(S)$ -comodule V there is an isomorphism: $A \square_H (H \square_{\psi(S)} V) \cong (A \square_H H) \square_{\psi(S)} V \cong A \square_{\psi(S)} V$, where the first isomorphism exists since A is coflat right H -comodule. Now, H is faithfully coflat right $\psi(S)$ comodule since A faithfully coflat as both right H and $\psi(S)$ -comodule (what follows from faithfully flatness of A as right B and S -module and the canonical isomorphisms: $can_B : A \otimes_B A \cong A \otimes H$ and $can_S : A \otimes_S A \cong A \otimes \psi(S)$). ■

Proposition 4.10

Let H be a flat R -Hopf algebra with bijective antipode. The map $\text{Quot}_{gen}(H) \ni I \mapsto I^{op} := S_H(I) \in \text{Quot}_{gen}(H^{op})$ is a bijection with inverse $\text{Quot}_{gen}(H^{op}) \ni J \mapsto J^{op} := S_{H^{op}}(J) \in$

$\text{Quot}_{\text{gen}}(H)$. The coideal $S_H(I)$ is right (left) admissible if and only if I is right (left) admissible.

Proof: The map $\text{Quot}_{\text{gen}}(H) \ni I \mapsto I^{\text{op}} := S_H(I) \in \text{Quot}_{\text{gen}}(H^{\text{op}})$ is bijection since the antipode S_H is bijective and the inverse $S_H^{-1} = S_{H^{\text{op}}}$.

The antipode defines an R -linear bijection $H/I \cong H/S_H(I)$ thus $H/S_H(I)$ is R -flat whenever H/I is. Let V be a left $H/S_H(I)$ -comodule. We make it a right H/I -comodule by $V \ni v \mapsto v_{(0)} \otimes S_H^{-1}(v_{(1)})$. Then we have

$$H \square_{H/S_H(I)} V \cong V \square_{H/I} H \\ \sum_i h_i \otimes v_i \mapsto \sum_i v_i \otimes h_i$$

It is natural in V , hence if H is right faithfully coflat over $H/S_H(I)$. This shows that $S_H(I)$ is right admissible if and only if I is left admissible. That I is right admissible if and only if $S_H(I)$ is left admissible is proved in the same way. ■

Now we present the proof of Theorem 4.7 which is due to Schauenburg.

Proof of Theorem 4.7: We let $(\varphi^{\text{op}}, \psi^{\text{op}})$ denote the Galois connection (4) for A^{op} H^{op} -comodule algebra instead of A – an H -comodule algebra.

Let $S \subseteq A$ be left admissible. Then $S^{\text{op}} \subseteq A^{\text{op}}$ is right admissible, thus $\psi^{\text{op}}(S^{\text{op}})$ is right admissible and by Proposition 4.9(1), $\varphi^{\text{op}}\psi^{\text{op}}(S^{\text{op}}) = S^{\text{op}}$. It follows that $S = (\varphi^{\text{op}}\psi^{\text{op}}(S^{\text{op}}))^{\text{op}} = \varphi(\psi^{\text{op}}(S^{\text{op}})^{\text{op}})$, by Lemma 4.8. Using Proposition 4.10 we conclude that $(\psi^{\text{op}}(S^{\text{op}}))^{\text{op}}$ is left admissible thus: $\psi(S) = \psi\varphi(\psi^{\text{op}}(S^{\text{op}})^{\text{op}}) = \psi^{\text{op}}(S^{\text{op}})^{\text{op}}$, hence $S = \varphi\psi(S)$ and $\psi(S)$ is left admissible.

We follow the argument of Schauenburg. Let I be right admissible. Then I^{op} is left admissible, so is $\varphi^{\text{op}}(H^{\text{op}}/I^{\text{op}}) = \varphi(H/I)^{\text{op}}$, thus $\varphi(H/I)$ is right admissible, and $H^{\text{op}}/I^{\text{op}} = \psi^{\text{op}}\varphi^{\text{op}}(H^{\text{op}}/I^{\text{op}}) = (\psi\varphi(H/I))^{\text{op}}$ hence $I^{\text{op}} = (\bar{\psi}\varphi(H/I))^{\text{op}}$ and thus $I = \bar{\psi}\varphi(H/I)$, i.e. $H/I = \psi\varphi(H/I)$. ■

5 Galois theory for Galois coextensions

In this section we describe the Galois theory for Galois coextensions. We begin with some basic definitions.

Definition 5.1

Let C be a coalgebra and H a Hopf algebra, both over a ring R . We call C an H -module coalgebra if it is an H -module such that the H -action $H \otimes C \rightarrow C$ is a coalgebra map:

$$\Delta_C(h \cdot c) = \Delta_C(h)\Delta_H(c), \quad \varepsilon_C(h \cdot c) = \varepsilon_C(h)\varepsilon_H(c).$$

Let $C^H := C/H^+C$ denote the invariant coalgebra. We call $C \rightarrow C^H$ an H -coextension.

Definition 5.2

An H -module coalgebra C is called an H -Galois coextension if the canonical map

$$\text{can}_H : H \otimes C \rightarrow C \square_{C^H} C, \quad h \otimes c \mapsto hc_{(1)} \otimes c_{(2)}$$

is a bijection, where C is considered as a left and right C^H -comodule in a standard way. More generally, if $K \in \text{cold}_l(H)$ is a left coideal then K^+C is a coideal (at least, when the base ring R is a field) and an H -submodule of C . The coextension $C \rightarrow C^K = C/K^+C$ is called Galois if the canonical map

$$\text{can}_K : K \otimes C \rightarrow C \square_{C^K} C, \quad k \otimes c \mapsto kc_{(1)} \otimes c_{(2)} \quad (6)$$

is a bijection.

Basic example of an H -module coalgebra is H itself. Then $H^H = R$ and $H \square_{H^H} H = H \otimes H$. The inverse of the canonical map is given by $\text{can}_H^{-1}(k \otimes h) = kS(h_{(1)}) \otimes h_{(2)}$.

Definition 5.3

Let C be an H -module coalgebra. We let $\text{Quot}(C) = \{C/I : I \text{ a coideal of } C\}$. It is a complete lattice. We let $\text{Quot}(C/C^H)$ denote the interval $\{Q \in \text{Quot}(C) : C^H \leq Q \leq C\}$ in $\text{Quot}(C)$.

Proposition 5.4

Let C be an H -module coalgebra over a field k . Then there exists a Galois connection:

$$\begin{array}{ccc} \text{Quot}(C/C^H) & \xrightleftharpoons{\quad} & \text{cold}_l(H) \\ & C/(I+k1_H)^+C \longleftarrow I & \end{array} \quad (7)$$

Proof: The supremum in $\text{cold}_l(H)$ is given by sum of submodules. Thus the lattice of left coideals is complete. Furthermore, if I is a right coideal then $I + k1_H$ is also a right coideal. It is enough to show that the map $\text{cold}_l(H) \ni I \mapsto I^+C \in \text{cold}(C)$ preserves all suprema when we restrict to right coideals which contain 1_H . Let $I_\alpha \in \text{Id}_{\text{gen}}(H)$, for $\alpha \in \Lambda$, then $(+_\alpha I_\alpha)^+ = +_\alpha(I_\alpha^+)$. The non trivial inclusion is $(+_\alpha I_\alpha)^+ \subseteq +_\alpha(I_\alpha^+)$. Let $k = \sum_\alpha k_\alpha \in (+_\alpha I_\alpha)^+$, i.e. $k_\alpha \in I_\alpha$ and $\sum_\alpha k_\alpha \in \ker \varepsilon$. Then $\sum_\alpha k_\alpha = \sum_\alpha (k_\alpha - \varepsilon(k_\alpha)1) + \sum_\alpha (\varepsilon(k_\alpha)1) = \sum_\alpha (k_\alpha - \varepsilon(k_\alpha)1)$. Now each $k_\alpha - \varepsilon(k_\alpha)1 \in I_\alpha^+$ and hence $k \in +_\alpha(I_\alpha^+)$. ■

Theorem 5.5

Let C be an H -module coalgebra over a field k with monomorphic canonical map can_H . Let K_1, K_2 be two left coideals of H such that both can_{K_1} and can_{K_2} are bijections. Then $K_1 = K_2$ whenever $C^{K_1} = C^{K_2}$.

Proof: We have the following commutative diagram:

$$\begin{array}{ccccc} & & K_1 \otimes C & & \\ & \swarrow \text{can}_{K_1} & \downarrow i_1 \otimes id & \searrow & \\ C \square_{C^{K_1}} C & \xrightarrow{\quad} & C \square_{C^H} C & \xleftarrow{\text{can}} & H \otimes C \\ & \nwarrow \text{can}_{K_2} & \uparrow i_2 \otimes id & \nearrow & \\ & & K_2 \otimes C & & \end{array}$$

It follows that $i_2 \otimes id \circ (\text{can}_{K_2} \circ \text{can}_{K_1}^{-1}) = i_1 \otimes id$ and thus $K_1 \subseteq K_2$; similarly $K_2 \subseteq K_1$. ■

Corollary 5.6

Let C be an H -coextension with monomorphic canonical map can_H . Then K - a left coideal of H such that $1_H \in K$ is a closed element of Galois connection (7) if C is K -Galois.

Proof: Let C be a K -Galois coextension, for some left coideal K of H such that $1_H \in K$ and let \tilde{K} be the smallest closed left coideal such that $K \subseteq \tilde{K}$. Then we have the commutative diagram:

$$\begin{array}{ccc}
 H \otimes C & \xrightarrow{can_H} & C \square_{C^H} C \\
 \uparrow \wr & & \uparrow \wr \\
 \tilde{K} \otimes C & \xrightarrow{can_{\tilde{K}}} & C \square_{C^{\tilde{K}}} C \\
 \uparrow \wr & & \uparrow \parallel \\
 K \otimes C & \xrightarrow[\simeq]{can_K} & C \square_{C^K} C
 \end{array}$$

From lower commutative square it follows that $can_{\tilde{K}}$ is onto, while from the upper that it is a monomorphism. The result follows now from the previous theorem. \blacksquare

6 Correspondence between left coideal subalgebras and generalised quotients

We show a new simple prove of Takeuchi correspondence between left coideal subalgebras and right H -module coalgebra quotients for a finite dimensional Hopf algebra. We also show that for arbitrary Hopf algebra H the generalised quotient Q is closed if and only if $H^{coQ} \subseteq H$ is Q -Galois. Similarly for a left coideal subalgebras: it is closed if and only if $H \rightarrow H^K$ is a K -Galois coextension.

Theorem 6.1

Let H be a flat Hopf algebra over a ring R with bijective antipode. The extension H/R is H -Galois and there exists a Galois connection:

$$\left\{ \begin{array}{l} K \subseteq H : K \text{ - left coideal subalgebra} \\ =: \text{Sub}_{gen}(H) \end{array} \right\} \begin{array}{c} \xrightarrow{\psi} \\ \xleftarrow{\varphi} \end{array} \left\{ \begin{array}{l} H/I : I \text{ - right ideal coideal} \\ =: \text{Quot}_{gen}(H) \end{array} \right\} \quad (8)$$

where $\varphi(Q) = H^{coQ}$, $\psi(K) = H/K^+H$ is a Galois connection which is a restriction of the Galois connection (4). Moreover, this Galois correspondence restricts to normal Hopf subalgebras and conormal Hopf quotients. The following holds:

- (1) $K \in \text{Sub}_{gen}(H)$ such, that H is (left, right) faithfully flat over K , is a closed element of the above Galois connection,
- (2) $Q \in \text{Quot}_{gen}(H)$ such, that H is (left, right) faithfully coflat over Q is a closed element of Galois connection (8),
- (3) if H is finite dimensional then φ and ψ are inverse bijections.

The Galois correspondence restricts to bijection between elements satisfying (1) and (2).

The existence of this Galois correspondence is proved in [Schauenburg, 1998], where the author notes that it is a folklore even in the case when R is a ring rather than just a field. Let us note that existence does not require H to have a bijective antipode. The points (1) and (2) follows from Theorem 4.7 (due to Schauenburg, see also [Schauenburg, 1998, Thm 3.10]), while point (3) follows from [Skryabin, 2007, Thm 6.1], where it is shown that if H is finite dimensional then it is free over every its right (or left) coideal subalgebra (see [Skryabin, 2007, Thm 6.6] and also [Schauenburg and Schneider, 2005, Cor. 3.3]). This theorem has a long history. The study of this correspondence, with Hopf algebraic method, goes back to Takeuchi [1972, 1979]. Then Masuoka proved (1) and (2) for Hopf algebras over a field (with bijective antipode). When the base ring is a field, Schneider [1993, Thm 1.4] proved that this bijection restricts to normal Hopf subalgebras and normal Hopf algebra quotients. For Hopf algebras over more general rings it was shown by Schauenburg [1998, Thm 3.10]. We can present a new simple proof of 6.1(3), which avoids Skryabin result.

Proof of Theorem 6.1(3): Whenever H is finite dimensional, for every Q the extension $H^{coQ} \subseteq H$ is Q -Galois by [Schauenburg and Schneider, 2005, Cor. 3.3]. Using Proposition 4.2 we get that the map φ is a monomorphism. To show that it is an isomorphism it is enough to prove that ψ is a monomorphism. We now want to consider H^* . To distinguish φ and ψ for H and H^* we will write φ_H and ψ_H considering (8) for H and φ_{H^*} and ψ_{H^*} considering H^* . It turns out that $(\psi_H(K))^* = \varphi_{H^*}(K^*)$. It can be easily shown that $can_{K^*} = (can_K)^* : H^* \otimes_{H^{coQ}} H^* \rightarrow H^* \otimes K^*$ under some natural identifications. Because H^* is finite dimensional as well it follows that can_{K^*} is an isomorphism, hence can_K is a bijection for every right coideal subalgebra K of H . Now the result follows from Theorem 5.5 and Proposition 2.2. ■

Theorem 6.2

Let H be a flat Hopf algebra over a ring R . Then

- (i) $Q \in \text{Quot}_{gen}(H)$ is a closed element of Galois connection (8) if and only if H/H^{coQ} is a Q -Galois extension,
- (ii) $K \in \text{Sub}_{gen}(H)$ is a closed element of the Galois connection (8) if and only if $H \rightarrow H^K$ is a K -Galois coextension.

Note that we do not assume that the antipode of H is bijective as it is done in Theorem 6.1. The flatness of H is only needed to show that if K is closed then $H \rightarrow H^K$ is a K -Galois coextension.

Proof: For the first part, it is enough to show that if Q is closed then $H^{coQ} \subseteq H$ is a Q -Galois (see Corollary 4.3). If Q is closed then $Q = H/(H^{coQ})^+ H$. One can show that for any $K \in \text{Sub}_{gen}(H)$ the following map is an isomorphism:

$$H \otimes_K H \rightarrow H \otimes H/K^+ H, \quad h \otimes_K h' \mapsto h h'_{(1)} \otimes \overline{h'_{(2)}} \quad (9)$$

Its inverse is given by $H \otimes H/K^+ H \ni h \otimes \overline{h'} \mapsto h S(h'_{(1)}) \otimes_K h'_{(2)} \in H \otimes_K H$ which is well defined since K is a left coideal. Plugging $K = H^{coQ}$ to equation (9) we observe that this map is the canonical map (1) associated to Q .

Now, if $H \rightarrow H/K^+H$ is a K -Galois coextension then it follows from Theorem 5.5, using the same argument as in Corollary 5.6, that K is a closed element. Now, let us assume that K is closed. Then $K = H^{coQ}$ for $Q = H/K^+H$. By [Schneider, 1992, Thm 1.4(1)] we have an isomorphism:

$$H^{coQ} \otimes H \longrightarrow H \square_Q H \quad k \otimes h \longmapsto kh_{(1)} \otimes h_{(2)} \quad (10)$$

with inverse $H \square_Q H \ni k \otimes h \longmapsto kS(h_{(1)}) \otimes h_{(2)} \in H^{coQ} \otimes H$. The above map is canonical map (6) since $K = H^{coQ}$ and $Q = H/K^+H$. ■

The above theorem gives an answer to the question when the bijection (8) holds without extra assumptions.

Corollary 6.3

The bijective correspondence (8) holds without flatness/coflatness assumptions if and only if for every $Q \in \text{Quot}_{gen}(H)$ H/H^{coQ} is Q -Galois extension and for every $K \in \text{Sub}_{gen}(H)$ H/H^K is a K -Galois coextension.

7 Crossed product extensions

Next we describe closed elements of the Galois connection (4) when A is a crossed product. An H -extension A/B (over a ring R) is called cleft if there exists a convolution invertible H -comodule map $\gamma : H \rightarrow A$. An H -extension $B \subseteq A$ has the normal basis property if and only if A is isomorphic to $B \otimes_k H$ as a left B -module and right H -comodule. If the base ring R is a field then the following conditions are equivalent:

- (i) A/B is a cleft extension,
- (ii) A/B is a Hopf–Galois extension with normal basis property,
- (iii) A is a crossed product of B and H , i.e. there exists an invertible cocycle $\sigma : H^{\otimes 2} \rightarrow A$ such that $A \cong B \#_{\sigma} H$ as H -comodule algebras.

For the proof see Doi and Takeuchi [1986] and Blattner and Montgomery [1989]. We refer to [Montgomery, 1993] for the theory of crossed products. Let us recall that the underlying R -module of $B \#_{\sigma} H$ is $B \otimes H$ while the multiplication is given by the formula:

$$a \#_{\sigma} h \cdot b \#_{\sigma} k := a(h_{(1)} \cdot b) \sigma(h_{(2)} \otimes k_{(1)}) \#_{\sigma} h_{(3)} k_{(2)}$$

Note that H acts on B in a compatible way (H measures B , see [Montgomery, 1993, Def. 7.1.1]). Let us note that the results presented below apply to finite Hopf–Galois extensions of division rings, since they are always crossed products by [Montgomery, 1993, Thm 8.3.7].

Theorem 7.1

Let A/B be an H -crossed product over a ring R with B a flat Mittag-Leffler R -module and H a flat R -module. Then the Galois correspondence (4) exists. Moreover, an element $Q \in \text{Quot}_{gen}(H)$ is closed if and only if the extension A/A^{coQ} is Q -Galois.

Proof: First of all, the Galois connection (4) exists, since we have a diagram:

$$\text{Sub}_{Alg}(B \#_{\sigma} H) \begin{array}{c} \xleftarrow{\omega} \\ \xrightarrow{\zeta} \end{array} \text{Sub}_{gen}(H) \begin{array}{c} \xleftarrow{\psi} \\ \xrightarrow{\varphi} \end{array} \text{Quot}_{gen}(H)$$

where (φ, ψ) is the Galois connection (8) and $\zeta(S) = B \otimes S$. The map ζ preserves all intersections and so it has a left adjoint ω . Then the Galois connection (4) exists and it has the form $(\zeta \circ \varphi, \psi \circ \omega)$, since by flatness of B we have $B \otimes H^{coQ} = (B \otimes H)^{coQ}$.

Let $Q \in \text{Quot}_{gen}(H)$ be a closed element of this Galois connection, i.e. $Q = \psi\omega\zeta\varphi(Q)$. Then it is closed element of (8), since it belongs to the image of ψ , and thus, by Theorem 6.2(i), the map: $can'_Q : H \otimes_{H^{coQ}} H \rightarrow H \otimes Q$ is an isomorphism. We have a commutative diagram:

$$\begin{array}{ccc} A \otimes_{A^{coQ}} A & \xrightarrow{can_Q} & A \otimes Q \\ \wr \downarrow & & \downarrow \parallel \\ B \otimes H \otimes_{H^{coQ}} H & \xrightarrow[id_B \otimes can'_Q]{\simeq} & B \otimes H \otimes Q \end{array}$$

Thus can_Q is an isomorphism. The converse follows from Corollary 4.3, since $can : A \otimes_B A \rightarrow A \otimes H$ is a bijection. \blacksquare

Now we formulate a criterion for closedness of subextensions of A/B which generalises 6.2(ii).

Theorem 7.2

Let A/B be an H -crossed product extension over a ring R with B a faithfully flat Mittag-Leffler module and H a flat (thus faithfully flat) R -module. Then a subalgebra $S \in \text{Sub}_{Alg}(A/B)$ is a closed element of the Galois connection (4) if and only if the canonical map:

$$can_S : S \otimes A \rightarrow A \square_{\psi(S)} H, \quad can_S(a \otimes b) = ab_{(1)} \otimes b_{(2)} \quad (11)$$

is an isomorphism.

Proof: First let us note that the map can_S is well defined since it is a composition of $S \otimes A \rightarrow A^{co\psi(S)} \otimes A$, induced by the inclusion $S \subseteq A^{co\psi(S)}$, with $A^{co\psi(S)} \otimes A \rightarrow A \square_{\psi(S)} H$ of [Schneider, 1992, Thm 1.4].

Now let us assume that can_S is an isomorphism. We let $K = H^{co\psi(S)}$. Since B is a flat R -module we have $(B \#_{\sigma} H)^{co\psi(S)} = B \#_{\sigma} K$. The following diagram commutes:

$$\begin{array}{ccc} S \otimes_B A & \xrightarrow{can_S} & A \square_{\psi(S)} H \\ \alpha \downarrow & & \downarrow \wr \\ B \#_{\sigma} K \otimes H & \xrightarrow{\beta} & B \otimes (H \square_{\psi(S)} H) \end{array} \quad (12)$$

where $\alpha : S \otimes_B A \rightarrow (B \#_{\sigma} K) \otimes_B (B \#_{\sigma} H) \rightarrow B \#_{\sigma} K \otimes H$ is given by

$$\alpha(a \otimes_B b \#_{\sigma} h) = (a \cdot b \#_{\sigma} 1_H) \otimes h \quad \text{for } a \in S \subseteq B \#_{\sigma} K, b \#_{\sigma} h \in A$$

It is well defined since K is a left comodule subalgebra of H . The second vertical map $(B \#_{\sigma} H) \square_{\psi(S)} H \rightarrow B \otimes (H \square_{\psi(S)} H)$ is the natural isomorphism, since B is flat over R . The map $\beta : B \#_{\sigma} K \otimes H \rightarrow B \otimes (H \square_{\psi(S)} H)$ is defined by $\beta(a \#_{\sigma} k \otimes h) = (a \#_{\sigma} k \cdot 1_{B \#_{\sigma} h_{(1)}}) \otimes h_{(2)}$. Note that $\beta = id_B \otimes can_K \circ \gamma$ where $\gamma : (B \#_{\sigma} K) \otimes H \rightarrow (B \#_{\sigma} K) \otimes H$, $\gamma(a \#_{\sigma} k \otimes h) :=$

$a\sigma(k_{(1)}, h_{(1)}) \otimes k_{(2)} \otimes h_{(2)}$ is an isomorphism, since σ is convolution invertible, while can_K denotes the canonical map (10). Furthermore, the map can_K is an isomorphism by Theorem 6.2(ii), since $K := H^{co\psi(S)}$. By commutativity of the above diagram it follows that α is an isomorphism. Now, let us consider the commutative diagram:

$$\begin{array}{ccc} S \otimes_B (B \#_{\sigma} H) & \xrightarrow{\alpha} & B \#_{\sigma} K \otimes H \\ \downarrow & \nearrow \simeq & \\ (B \#_{\sigma} K) \otimes_B (B \#_{\sigma} H) & & \end{array}$$

Thus $S = B \#_{\sigma} K$ is indeed closed, since α is an isomorphism and $B \#_{\sigma} H$ is a faithfully flat B -module under the assumptions made.

If S is closed then $S = A^{co\psi(S)} = B \#_{\sigma}(H^{co\psi(S)})$, since B is a flat R -module. One easily checks that α is an isomorphism with inverse: $\alpha^{-1}(a \#_{\sigma} k \otimes h) = (a \#_{\sigma} k) \otimes_B (1_B \#_{\sigma} h)$. The left coideal subalgebra $K = H^{co\psi(S)}$ is a closed element of (8) hence by Theorem 6.2(ii) can_K is an isomorphism and thus β is an isomorphism. It follows from (12) that can_S is an isomorphism as well. ■

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